Boundary element formulation of axisymmetric problems for an elastic halfspace

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elastic halfspace, given in terms of integrals of the Lipschitz…Hankel type, that satis“es in advance the boundary condition of zero traction on the free surface and the decay of displacements in the far “eld. Explicit equations for post-processing the results at internal points are provided, as well as adequate numerical schemes to evaluate the boundary integrals arising in the method. This formulation can be easily implemented in existing BE computational codes for axisymmetric fullspace problems, requiring only a few modi“cations. Numerical results are provided to validate the proposed formulation.

1. Introduction

The axisymmetric formulation in classical elasticity is useful for the analysis of problems in geomechanics [1,2], as well as contact problems for cylinders, spheres and circular plates [3..8]. Other applications involve the study of fracture mechanics phenomena and inclusions [5,9..11].

In particular, the BE method is advantageous for axisymmetric problems, since it reduces the analysis of the three-dimensional domain to a one-dimensional mesh discretization requiring only the evaluation of linear integrals. However, the fundamental solutions involved are more complex, requiring special considerations on their manipulation and integration to correctly evaluate the in”uence coef“cients arising in the boundary integral equations. Extensive surveys on the existing axisymmetric fundamental solutions are given by Wang and Liao [12,13], Wang et al. [14] and Wideberg and Benitez [15].

The BE method for axisymmetric elasticity was “rst formulated by Cruse et al. [16], using the fullspace fundamental solution derived by Kermanidis [17]. Several contributions to the formulation of the axisymmetric problem may be cited, such as the expansion of non-symmetric boundary conditions by Fourier series suggested by Mayr [18] and Rizzo and Shippy [19,20

the number of unknown functions can be reduced from 12 to 6. These functions can be determined using (i) the compatibility conditions of displacements at the interface

u_i^{I}ðr

$$u_{zr}^{nd} ¼ \frac{1}{\quad}$$

where G_E and $\overline{G}_E$ are portions of the circumference of radius E, as depicted in Fig. 4.

$z \neq 0$. On the other hand, for $z=0$ the traction term t^{nd}_{ij} vanishes and only u^{nd}_{ij} presents singularity.

In this section, the integration cases identified for G^t_{ia} and $\hat{H}^t_{ia}$ are grouped according to the position of $P(x,z')$ in relation to the part of the boundary along which the integration is carried out as well as to the axis of symmetry. The numerical schemes employed in this work to evaluate regular integrals, weakly singular integrals of logarithmic terms and the finite part of singular integrals of order $1/r$ are briefly presented in Appendix C.

4.1. Case 1: $P(x,z') \notin \Gamma_t$

If the point $P(x,z')$ does not belong to the portion of the boundary being integrated, as illustrated in Fig. 5, then $r > 0$ and accordingly both G^t_{ia} and $\hat{H}^t_{ia}$ expressed in Eqs. (56) and (57) are regular and can be evaluated by the Gauss–Legendre quadrature rule [65]. For each portion of the boundary, these integrals can be rewritten in terms of the natural coordinate $\xi \in [-1,1]$ as

$$G^t_{ia} = 2\pi \int_{-1}^{1} u^{nh}_{ij} N_a(\xi) r(\xi) J(\xi) d\xi \quad (58)$$

$$\hat{H}^t_{ia} = 2\pi \int_{-1}^{1} t^{nh}_{ij} N_a(\xi) r(\xi) J(\xi) d\xi \quad (59)$$

where $J(\xi)$

Gauss–Legendre quadrature rule while the second integral has logarithmic singularity and should be evaluated by the weighted logarithmic Gauss quadrature rule [65]. Although the complete elliptic integral $E(m)$ presents no singularity, its nonsingular approximation also includes logarithmic terms which are also isolated to enhance numerical convergence.

Owing to Eqs. (60)–(63), $\hat{H}^t_{ia}$ as given by Eq. (57) also includes logarithmic terms in $t^{nf}_{K_{ij}}K(m)$ and $t^{nf}_{E_{ij}}E_2(m)\ln \bar{m}$. Thus, their integration can be carried out by means of a regular integral and a weakly logarithmic singular integral, similar to the procedure proposed for G^t_{ia}. On the other hand, the integral of $t^{nf}_{E_{ij}}E_1(m)$ exists only in terms of the finite part, to be numerically evaluated by the scheme proposed by Dumont and Souza [67] for singular integrals in terms of order $1/r$ over a curved boundary. This procedure employs the Gauss–Legendre quadrature and an additional correction term, as summarized in Appendix C. Hence, $\hat{H}^t_{ia}$ can be given by

$$\hat{H}^t_{ia} = 2\pi\int_{-1}^{1}\left\{ t^{nf}_{K_{ij}}\left[K_1 + K_2 \ln\frac{(1-\xi^0\xi)^2}{4\bar{m}}\right] + t^{nf}_{E_{ij}}\left[E_1 + E_2 \ln\frac{(1-\xi^0\xi)^2}{4\bar{m}}\right]\right\} N_a(\xi)\, r(\xi)\, J(\xi)\, d\xi$$
$$- 8\pi\int_{0}^{1}\left(t^{nf}_{K_{ij}}K_2 + t^{nf}_{E_{ij}}E_2\right)\ln\xi\, N_a(\xi)\, r(\xi)\, J(\xi)\, d\xi$$
$$- 2\pi\xi^0 t^{nf}_{E_{ij}}\, r\, N_a(\xi)\, r(\xi)\big|_{\xi=\xi^0}\left\{\ln\left|2J\right|_{\xi=\xi^0} - \sum_{m=1}^{n_g}\frac{w^g_m}{1+\xi^g_m}\right\} \qquad (66)$$

where ξ^0 is the value of ξ at the singularity point (which is either -1 or 1) and ξ^g_m and w^g_m are the abscissae and weights of the Gauss–Legendre quadrature scheme for n_g points. In the above equation, the limit of $t^{nf}_{E_{ij}}\, r\, N_a(\xi)\, r(\xi)$ for $\xi \to \xi^0$ is difficult to obtain analytically and has been evaluated numerically by extrapolation.

Alternatively, the elements of the matrix $\hat{H}^t_{ia}$ that require the evaluation of singular integrals can be obtained indirectly by

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The complete elliptic integrals can be approximated as in Eqs. (62) and (63). Similar to the procedure presented for Case 2.1, one arrives at

$$G^{t}_{la} = 2\pi \int_{1}^{1} u$$

$$\frac{2ck^5}{p\bar{k}^4 A_1^5}\left[6(k^2-2)\frac{c^2k^2}{A_1^2}-\frac{8k^4}{\bar{k}^2}+23\right]E(m) \quad (A.11)$$

$$I_{113} = \frac{2ck^3}{p\bar{k}^2 A_1^5}\left[3(k^2-2)+\frac{2c^2k^2}{A_1^2}-\frac{2k^4}{\bar{k}^2}+3\right]K(m)$$
$$-\frac{2ck^3}{p\bar{k}^4 A_1^5}\left[6(k^4+\bar{k}^2)+\frac{c^2k^2(k^2-2)(\bar{k}^2+8k^4)}{\bar{k}^2 A_1^2}\right]E(m) \quad (A.12)$$

ū^nf_zzr ¼ 1/(8pð1 nÞ) f P2Ī011 þ 9z9Ī012 g ð8:6Þ

ū^nf_zzz ¼ 1/(8pð1 nÞ) f signðzÞ2P1Ī001 zĪ002 g ð8:7Þ

ū^nd_rrr ¼ 1/(8pð1 nÞ) 1/x ½ P7Î110 þ P5 9z9Î111 2zz0Î112 þ P6Î011
signðzÞðP5z0 þ 3zÞÎ012 þ 2zz0Î013 o ð8:8Þ

ū^nd_rrz ¼ 1/(8pð1 nÞ) 1/x ½ signðzÞ4P1P2Î100 P5zÎ101
signðzÞ2zz0Î102 o signðzÞ2P3Î001 þðP5z0 3zÞÎ002
þ signðzÞ2zz0Î003 ð8:9Þ

ū^nd_rzr ¼ 1/(8pð1 nÞ) f signðzÞ2P1Î111 þðP5z0 þ zÞÎ112 signðzÞ2zz0Î113 g
ð8:10Þ

ū^nd_rzz ¼ 1/(8pð1 nÞ) fP2Î101 signðzÞðP5z0 zÞÎ102 2zz0Î103 g ð8:11Þ

ū^nd_zzr ¼ 1/(8pð1 nÞ) fP2Î011 þ signðzÞðP5z0 zÞÎ012 2zz0Î013 g ð8:12Þ

ū^nd_zzz ¼ 1/(8pð1 nÞ) f signðzÞ2P1Î001 ð P5z0 þ zÞÎ002
2signðzÞzz0Î003 g ð8:13Þ

s̄^nf_rrrr ¼ m/(4pð1 nÞ) 1/x ½P4Ī101 9z9Ī102 þ 1/(xr) ½ P5Ī110 þ 9z9Ī111
þ 1/r ½P4Ī011 9z9Ī012 3Ī002 9z9Ī003 ð8:14Þ

s̄^nf_rrrz ¼ m/(4pð1 nÞ) 1/x ½signðzÞ2P1Ī111 zĪ112 signðzÞ2Ī012 þ zĪ013
ð8:15Þ

s̄^nf_rrzz ¼ m/(4pð1 nÞ) 1/x ½ P2Ī101 þ 9z9Ī102 þ Ī002 9z9Ī003 ð8:16Þ

s̄^nf_rzrr ¼ m/(4pð1 nÞ) 1/r ½ signðzÞ2P1Ī111 þ zĪ112 þ signðzÞ2Ī102 zĪ103
ð8:17Þ

s̄^nf_rzrz ¼ m/(4pð1 nÞ) fĪ112 9z9Ī113 g ð8:18Þ

s̄^nf_rzzz ¼ mzĪ103/(4pð1 nÞ) ð8:19Þ

s̄^nf_zzrr ¼ m/(4pð1 nÞ) 1/r ½ P2Ī011 þ 9z9Ī012 þ Ī002 9z9Ī003 ð8:20Þ

s̄^nf_zzrz ¼ mzĪ013/(4pð1 nÞ) ð8:21Þ

s̄^nf_zzzz ¼ m/(4pð1 nÞ) ... 002 þ 9z9 ... 003 g ð8:22Þ

s̄^nd_rrrr ¼ m/(4pð1 nÞ) 1/x ½P6Î101 signðzÞðP5z þ 3z0ÞÎ102
þ 2zz0Î103 þ 1/(xr) ½ P7Î110 2zz0Î112 þ P5 9z9Î111
þ 1/r ½P6Î011 signðzÞðP5z0 þ 3zÞÎ012 þ 2zz0Î013
o
5Î002 þ 39z9Î003 2zz0Î004 ð8:23Þ

s̄^nd_rrrz ¼ m/(4pð1 nÞ) 1/x ½ signðzÞ2P1Î111 þðP5z þ z0ÞÎ112
signðzÞ2zz0Î113 o þ signðzÞ2Î012 ð 3z þ z0ÞÎ013
signðzÞ2zz0Î014 ð8:24Þ

where

$$J(\xi) = \sqrt{\left(\frac{dr}{d\xi}\right)^2 + \left(\frac{dz}{d\xi}\right)^2} \qquad (C.2)$$

is the Jacobian transformation between the global and natural coordinate systems. The coefficients ξ_m^g and w_m^g are the abscissas and weights of the Gauss–Legendre quadrature rule for n_g points within the interval $(-1;1)$, which suffice to exactly evaluate the integral of a polynomial of order $2n_g - 1$.

C.2. Weakly singular integral of logarithmic terms

Let $f(r,z)$ be a regular function and

where $\overline{m}$ is given by Eq. (C.8) and

$$E_1(\overline{m}) = 1 + a_1\overline{m} + \dots + a_4\overline{m}^4$$

$$E_2(\overline{m}) = b_1\overline{m} + \dots + b_4\overline{m}^4 \quad (C.19)$$

are the polynomials whose coefficients are given by

$$a_1 = 0.44325141463, \quad b_1 = 0.24998368310$$

$$a_2 = 0.06260601220, \quad b_2 = 0.09200180037$$

$$a_3 = 0.04757383546, \quad b_3 = 0.04069697526$$

$$a_4 = 0.01736506451, \quad b_4 = 0.00526449639 \quad (C.20)$$

The polynomial approximation of $E(m)$ presents no singularity, since $E_2(\overline{m})$ has no free coefficients, according to Eq. (C.18). However, the presence of $\ln \overline{m}$ causes the integrand of Eq. (C.17) to be non-analytical, which requires a special numerical treatment.

In a manner similar to that used in the previous section, the following expression can be obtained for the numerical evaluation of the weakly singular integral given by Eq. (C.17)

$$\int_\Gamma f(r,z)E(m)\,d\Gamma$$

$$\approx \sum_{m=1}^{n_g} f(\xi)\left[E_1(\overline{m}) + 2E_2(\overline{m})\ln\left(\frac{1}{2}+\frac{\xi'\xi}{\overline{m}}\right)\right] J(\xi)\Big|_{\xi=\xi_m^g} w_m^g \quad (C.21)$$

$$4\sum_{m=1}^{n_l} f(\overline{\xi})E_2(\overline{m})\ln\overline{\xi}\,J(\overline{\xi})\Big|_{\overline{\xi}=\overline{\xi}_m} w_m^l \quad (C.22)$$

for $\overline{\xi}$ given by Eq. (C.6).

C.3. Cauchy principal value of the singular integral of order $1/r$

Let $f(r,z)$ be a regular function and $r(r,z)$ the distance between the points $P(x,z^0)$ and $Q(r,z)$ on the boundary $\Gamma(r,z)$. The strongly singular integral

$$\int_\Gamma \frac{f(r,z)}{r(r,z)}\,d\Gamma \quad (C.23)$$

has to be evaluated for the case $r(-1) = 0$ or $r(1) = 0$. This integral may be obtained as a sum of a Cauchy principal value and a discontinuous term as

$$\int_\Gamma \frac{f(r,z)}{r(r,z)}\,d\Gamma = PV\int_\Gamma \frac{f(r,z)}{r(r,z)}\,d\Gamma + c \quad (C.24)$$

The evaluation of the discontinuous term c of the strongly singular integrals appearing in the boundary element formulations is addressed in Section 3.2.

The Cauchy principal value is best evaluated in terms of two finite-part integrals, denoted by $\oint$, for the boundary segments adjacent to the singularity point $r(r,z) = 0$.

In what follows, the integration scheme proposed by Dumont and Souza [67] is used. Using the notation of Eq. (C.6), the regular function can be expanded as a Taylor series to obtain the following normalized integral of Eq. (C.23) over the curved boundary Γ

$$\int_\Gamma \frac{f(r,z)}{r(r,z)}\,d\Gamma = \xi'[\tilde{f}(\xi)\ln|r'|]_{\xi=\xi'} + \int_{-1}^{1}\frac{f(\xi)}{r(\xi)}J(\xi)\,d\xi \quad (C.25)$$

The resulting quadrature rule for evaluating Cauchy's principal value of the strongly singular integral of (C.23) is given by

$$\int_\Gamma \frac{f(r,z)}{r(r,z)}\,d\Gamma \approx \sum_{m=1}^{n_g}\frac{f(\xi)}{r(\xi)}J(\xi)\Big|_{\xi=\xi_m^g} w_m^g + \xi'[\tilde{f}(\xi)]_{\xi=\xi'}\left\{\ln|r'|_{\xi=\xi'} - \sum_{m=1}^{n_g}\frac{w_m^g}{1-\xi_m^g}\right\} \quad (C.26)$$

where

$$[\tilde{f}(\xi)]_{\xi=\xi'} = [f(\xi)J(\xi)]_{\xi=\xi'} \quad (C.27)$$

The above scheme, that employs the Gauss–Legendre quadrature rule and an additional correction term, evaluates exactly this integral for a polynomial function of order $2 n_g$. Other numerical integration schemes for the strongly singular integral can be used [72,73].

References

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